STABILITY OF SPHERICAL STELLAR SYSTEMS BY SYMPLECTIC METHOD : NUMERICAL TEST

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Abstract. We present a numerical analysis of a class of radial-anisotropic collisonless spherical stellar systems in order to understand the physical origin of the radial-orbit instability. This work is motivated by new analytical results based on the symplectic formulation of a generalised energy variationnal technique. We have obtained a first confirmation that the stability of such systems is governed by their aptitude to receive a certain class of perturbations.

1 Introduction

The study of the stability of equilibrium self-gravitating systems is a key-problem in stellar dynamics. It is motivated by three general problems which are modelling, formation and response.

Actual observations enable us to measure physical characteristics of gravitating stellar systems. It is now possible to confront theory with numerical experiences. The application of modern numerical methods allows a wide range of model simulations. However, the problem remains to choose the 'good' or 'real' model between all possible models. It is natural to assume that the stability of the system can be used to discriminate between choices. A corresponding study is, however, not straightforward, the analytical structure of the inhomogeneous and, in general, anisotropic case make the classical normal mode analysis very unadapted. The only real way to deal with this problem is to use variational energy techniques. For isotropic spherical cases, it is possible to use a variational principle and to obtain later the stability of systems with a monotoneously decreasing distribution function. An extension of these techniques was made in the case of anisotropic, radially perturbed systems (Sygnet et al. 1984, Kandrup and Sygnet 1985). However, the general anisotropic case is not totally tractable by this 'Hermitian' method.

A global analytical answer will perhaps come from the application of symplectic methods. Introduced by the pioneering work of Bartholomew (1971), revisited by Kandrup (1991) and improved by Perez and Aly (1993 hereafter Paper I), this approach gives a general stability criterion for any system. Applying this result, we showed (Paper I) that the stability of an anisotropic spherical system is directly related to the so-called 'preserving' perturbation. It has been shown that systems which can only be perturbed by preserving perturbations (i.e. isotropic systems for example) are stable. On the contrary, it is impossible to generate non vanishing preserving perturbations for systems with stars only on radial orbits. These systems are known to suffer 'radial orbit' instability with a singular form for their distribution function. One of the objective of this paper is to bring a numerical response for the intermediate case (Perez et al. 1993).

To make our experiences, we need two kinds of numerical tools. First, we need a relaxed initial condition generator (position and velocity). This set of coordinates must have several important properties. The systems they fit must be relaxed, in the equilibrium case, the virial ratio $\kappa = E_{cin}/2E_{pot} = -1$. Moreover, physical parameters of the modelised system (total mass, size, dynamical time, number of components, $\cdots$) must be ajustable to describe the largest class of possible systems. Finally, a large domain of the system's anisotropy must be considered. In this way, we have choosen the

Ossipkov-Merrit Algorithm (see Binney and Tremaine 1986 hereafter BT). This method based on generalised Abel inversion, consists in the distortion of a known isotropic model (here a polytropic one) into an anisotropic system with essentially one control parameter, the anisotropy radius r_a.

Secondly, the dynamical evolution of these initial conditions submitted to their own gavitationnal potential is assured by a direct N-Body code with softening parameter (see this volume the paper of H. Scholl & J.-M. Alimi). This code has been developped on a Connection Machine (Massivelly parallel supercomputer) (Alimi & Scholl 1993). This code and this machine allow us to consider a large range of several sensible parameters like number of particles, time resolution, conservation of energy, and number of experiences.

The architecture of this paper is the following : In the first part (section 2), we present briefly the main analytical results presented in Paper I. In a second part(section 3), we present the whole initial condition algorithm and the dynamical code. In the two last sections we present some preliminary results and their discussions.

2 Analytical Results

2.1 The Symplectic Formulation and Stability Criteria

As it was shown by Morrison 1982 the standard Vlasov equation (dynamical equation of collisonless systems) can be expressed using functionals. Indeed, if F is any functional of the phase space, the motion evolution equation can be written

$$\dot{F}[f] \;=\; \int f[\frac{\delta F}{\delta f},\frac{\delta H}{\delta f}]d\Gamma \;:=\; \{F,H\}[f] \tag{1}$$

where $[,]$ denotes the standard Poisson bracket in the canonical conjugated variables p and q. $\{,\}$ have all the properties of a Lie bracket and $\delta/\delta f$ meaning a functional derivative operation. Due to the fact that any physical perturbation of any initial state f_o can be described by a generator g which is an hamiltonian-like function representing the canonical transformation effected, (e.g. Bartolomew 1971), we can find (Paper I) a general Taylor expansion for any functional F during this perturbation

$$\begin{aligned}\hat{F}[f_o] &= F[f_0] - \{G,F\} + \frac{1}{2}\{G,\{G,F\}\} - \frac{1}{3!}\{G,\{G,\{G,F\}\}\} + \cdots \\ &= (e^{\{\,.\,,G\,\}}F)[f_o] \;=\; F[e^{[\,g\,,\,.\,]}f_o] \end{aligned} \tag{2}$$

where G is such that $\delta G/\delta f = g$ is the Functional generator. It is important to note the development (2) is true for any functional, so we can apply it to all functionals intervening in our problem as energy, entropy or more complicated.

Applying (2) in the special case of the total energy of the system

$$H[f] \;=\; \int d\Gamma \frac{p^2}{2m} f(q,p,t) \;-\; \frac{Gm^2}{2} \int d\Gamma \int d\Gamma' \frac{f(q,p,t)f(q',p',t)}{\mid q-q' \mid} \tag{3}$$

and choosing for f_o a steady state, the first order energy variation is clearly vanishing, and the second order can be written

$$H^{(2)}[f_o] \;=\; -\frac{1}{2}\int [g,E][g,f_o] d\Gamma \;-\; \frac{Gm^2}{2}\int d\Gamma \int d\Gamma' \frac{[g,f_o].[g',f_o']}{\mid q-q' \mid} \tag{4}$$

where E, the functional derivative of $H[f]$, is the single-particle energy. The stability of the system against some perturbations generated by some g, can now be investigated in the anisotropic inhomogeneous case by the study of the sign of $H^{(2)}[f_o]$. However, this development gives us only a sufficient condition for stability. Indeed, we cannot make like in the paper of Laval , Mercier , Pellat (1965) a connection between $H^{(2)}[f_o]$ and some definite inner product in order to have a general energy principle. As it is quoted in Larsson 1991, the inner-product related to our general problem is indefinite. In this case, relations like Schwartz's inequality fail and do not allow us to connect our energy variation to some dynamical variable to diagnostic linear instabilities.

As it is quoted in BT, stability problems of a collisionless stellar system and of a gas volume in gravitational interaction are closely related. In fact, in many cases, the hydrodynamic problem (when it is identifiable) is simpler than its stellar analog case. Indeed, the hydrodynamic problem is three-dimensional while stellar systems have six degrees of freedom. Hence a technique to simplify the study of stellar systems consists to average over velocities, when it is possible, to deal with the hydrodynamic counterpart of the problem. One of the most important isotropic results, sometimes called

Antonov-Lebovitz Theorem(ALT) (see BT), is based on this technique and assurse the stability of $f_o(E)$'s systems against non-radial perturbations.

In a recent paper (Aly and Perez, 1992), we present a new demonstration of this important result. This is the combination of this new method and the symplectic approach of the stability criterion which allow us to obtain our result on the stability of spherical stellar systems.

2.2 Preserving Perturbation

A stationary spherical system has a distribution function which depends only on the energy E and on the squared norm L^2 of the angular momentum L. $f_o = f_o(E, L^2)$. We assume here that

$$f_E \;:=\; \frac{\partial f_0}{\partial E} \;\leq\; 0 \text{ and } f_{L^2} \;:=\; \frac{\partial f_0}{\partial L^2} \;\leq\; 0 \qquad (5)$$

We want to consider the stability of such an equilibrium with respect to the class of preserving perturbations, which are generated by all the functions g satisfying $[g, L^2] = 0$ This class of perturbations are quite general, indeed, in the $f(E)$ isotropic case all perturbations are preserving. In this anisotropic spherical case all $g(L^2, L_x, L_y, L_z)$ generate preserving perturbations like in particular spherically symetrics ones. In such case one can split the perturbation f_1 in two parts with one invariant against rotation

$$\begin{aligned} f_1 &= \overline{f_1} + \delta f_1 \\ \overline{f_1}(q,p) &= \int f_1(R(q), R(p))dR \text{ and } \int \delta f_1(R(q), R(p))dR = 0 \end{aligned}$$

where the averaging is made over all possible rotations R. Hence, one can show that the second order variation of the energy splits in $H^{(2)}[f_1] = H^{(2)}[\overline{f_1}] + H^{(2)}[\delta f_1]$. The first part being positive (Kandrup and Sygnet 1985) we study the second non-radial part and show (using a revisited proof of ALT)

$$H^{(2)}[\delta f_1] \;\geq\; \frac{1}{2}\int \frac{(\delta\rho_1)^2}{\mid f_E \mid} dr \;-\; \frac{Gm^2}{2}\int dr \int dr' \frac{\delta\rho_1 \delta\rho_1'}{\mid q - q' \mid} \geq 0 \qquad (6)$$

where the non radial part of the perturbed density $\delta\rho_1$ is directly obtained from δf_1 by a velocity averaging. This result assures the stability of any anisotropic spherical stellar system - having the properties (5) - against preserving perturbations. More precisely we can say that if a g-generated perturbation makes the system unstable, then $[g, L^2] \neq 0$.

As we said before, in the isotropic case all perturbations are preserving. Moreover, in the purely radial orbit case where $f_o = \phi(E)\delta(L^2)$, Paper I shows the absence of such pertubations and exposes the instability case. It has no sense to speak about a number of preserving perturbations that can affect a given system. Indeed, any perturbation set is functional with an infinite dimension. But, we can naturally make the conjecture that the stability of such systems is closely related to the value of the $[g, L^2]$ real random variable. This is the purpose of our tests explained in the following sections.

3 Numerical techniques

3.1 The initial condition generator

In most papers presenting some numericals results about the stability of spherical systems, the main objectif consists to show the radial orbit instability (Merritt and Aguilar 1985, Barnes et al.1986, Palmer and Papaloizou 1987,1988, Weinberg 1990, Saha,1991). This mechanism is roughly explained by Antonov (1973) but it is not yet well understood. The best classic explanation is given by an empiric formula (Fridmann and Polyachenko 1986) which states that a spherical stellar system is unstable if

$$\tau_s \ := \ \frac{K_r}{2K_t} \ > \ 1,7 \pm 0.2 \tag{7}$$

K_r and K_t denoting respectively the kinetic energy of stars on radial and transversal orbits. Most of previous works take for starting point a quantity directly related to τ_s. It is the case if one considers a distribution function of the form $f_o \approx E^n L^{2m}$, or if one takes a special form for the velocity dispersion tensor. However, in this optic, one looses all information about the physical parameters of the system, and may be, one can speak about unreallistic system.

Our choice is different and consist to study a fine aspect of the largest possible class of physical systems, taking into account the radial orbit instability as a physical fact. The Ossipkov-Merritt algorithm is well adapted to this kind of situation. This algoritm (e.g. Osipkov 1979; Merritt 1985a,b) is built on the Abel inversion technique, and generates an anisotropic system from a controlled deformation of an isotropic one. Starting with a known isotropic gravitational potential $\psi_{iso}(r)$, we can, obtain via Poisson equation, the associated density $\rho_{iso}(r)$. The Ossipkov-Merritt algorithm consists then to deform this density in the following way

$$\rho_{ani}(r) \;:=\; \left(1+\frac{r^2}{r_a^2}\right)\rho_{iso}(r) \tag{8}$$

where the parameter r_a , called anisotropic radius, controls the deformation. In these conditions, we can form a class of anisotropic distributions functions which depend both on E and L^2 through the variable

$$Q \;:=\; E+\frac{L^2}{2r_a^2} \tag{9}$$

and which can be calculated by the following relation

$$f_o(Q) \;=\; \frac{\sqrt{2}}{4\pi^2}\frac{d}{dQ}\int_Q^0 \frac{d\psi_{iso}}{\sqrt{\psi_{iso}-Q}}\frac{d\rho_{ani}}{d\psi_{iso}} \tag{10}$$

The velocity anisotropy at any radius r of this model is given by

$$\frac{\sigma_r^2}{\sigma_t^2} \;:=\; \frac{< v_r^2 >}{\frac{1}{2} < v_t^2 >} \;=\; 1+\frac{r^2}{r_a^2} \tag{11}$$

Anisotropy depends only on r and r_a, this means that the model is always isotropic in the center and becomes anisotropic outwards.

We have chosen for the isotropic potential, the polytropic model, solution of the Lame-Emden differential equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) \;=\; (-1)^n 4\pi G c_n \psi^n \tag{12}$$

$$c_n \;=\; \frac{(2\pi)^{\frac{3}{2}}\Gamma(n-\frac{1}{2})}{\Gamma(n+1)}$$

The potential range varies from a free parameter $\psi(0)<0$ to 0, and as there should be no gravitationnal force at the center of the system, $d\psi/dr=0$ at $r=0$. To keep the mass finite, the polytropic index n must be ≤ 5 and for a $\psi-$convergence $n>\frac{1}{2}$. Tuning n and $\psi(0)$, we can modify all the physical parameters of the stellar system (mass, dynamical time, size, $\cdots$). Different models are presented below

Table 1 : Physicals parameters

n	r_a	$Mass$	$Size$	T_d	$\psi(0)$	τ_s	n	r_a	$Mass$	$Size$	T_d	$\psi(0)$	τ_s
	10	5.65	2.34	1.93	1.62	1.02		10	6.04	5.33	1.07	1.3	1.00
	2	4.61	2.20	1.91	1.43	0.84		2	2.81	4.38	1.16	1.07	0.78
2	1.5	3.75	2.10	2.00	1.30	0.82	3	1.5	2.75	4.22	1.13	1.03	0.78
	1.2	3.67	2.05	1.99	1.24	0.81		1.2	3.25	4.18	1.08	1.02	0.82
	1	3.79	2.01	1.95	1.20	0.88		1	4.38	4.22	1.01	1.03	0.95
	10	4.75	13.75	0.74	1.15	0.93		10	3.66	29.95	0.84	1.09	1.16
	2	2.81	11.13	0.64	1.00	0.78		2	3.34	25.05	0.40	.985	0.77
4	1.5	2.87	10.64	0.61	0.97	0.79	4.5	1.5	4.11	24.35	0.36	0.97	1.06
	1.2	5.41	11.13	0.52	1.00	0.87		1.2	4.47	23.43	0.35	0.95	1.13
	1	4.11	10.30	0.56	0.95	0.93		1	9.62	25.05	0.29	.981	1.01

All this models are virial-relaxed, the adjustment to the relaxation is made by a $\psi(0)$ tuning. The corresponding distribution functions are plotted below in same normalised frame (for the same value of r_a)

Figure 1 : Phase space Distribution functions

An important physical limitation of this model can be seen on this figures. Each value of polytropic index has a critical r_a. For higher anisotropic systems, the distribution functions are negative in some region of the phase space. This is an illustration of the fact that an arbitrary spherical mass distribution cannot always be reproduced by radial orbits. Nevertheless our initial condition generator is made robust to this problem and gives more anisotropic models. But in this case we can't have the real distribution function. At this moment it is important to remark that many authors haven't such scruple and push their models in the far anisotropy.

Using this distribution function it is now possible to generate N position-velocity bodies.

3.2 Dynamical Evolution

For a complete analysis of our simulations we built several interpretation tools

- An axial ratio calculator. Taking a set of positions (the same work is possible for velocities) we compute the inertial matrix in the $(0, x, y, z)$ frame. We diagonalize it to obtain its eigenvalues $\lambda_1 > \lambda_2 > lambda_3$ and finally the two axial ratios

$$a = \frac{\lambda_1}{\lambda_2} \geq 1 \text{ and } b = \frac{\lambda_3}{\lambda_2} \leq 1 \tag{13}$$

  The calculation of the axial ratio at each step of evolution giving the evolution of the global system. In the initial spherical case, all $\lambda = 1$. This tool is well-known in this context (see Palmer and Papaloizou,1987,1988, Binney and Aguilar,1985)

- A distribution funtion reconstructor. To follow the dynamical evolution of the distribution function we have built a code which calculates the energy and squared angular momentum and evaluates the distribution function in the $E - L^2$ plane.

- The preservant tool. In view to test our analytical conjecture, we elaborate a code which calculates the poisson bracket between the peculiar displacement generator and the squared angular momentum of each particle. This is possible because of the relation

$$dr_i \;;= \;\frac{\partial g_i}{\partial p_i} \text{ and } dp_i \;;= \; -\frac{\partial g_i}{\partial r_i} \tag{14}$$

The statistical study of the random variable $[g, L^2]$ is the key point for the preservance.

We have analysed three kinds of models with different physical parameters. Model (1) with $n = 2$ and $r_a = 0.5$, have a parameter stablity $\tau_s = 1.75$; (2) and (3) respectively $n = 4.5, r_a = 1.2$ and $n = 3, r_a = 10$ are physically different but they are both stable (in prevision). The dynamical experience allows us to produce the following figures

Figure 2 : *Axial ratio analyse*

Figure 3 : $[g, L^2]$ *histogram*

In figure 2 we have plotted the evolution of the star distribution in the space. All initial models are spherical and their two axial ratios are 1. The distributions of (2) and (3) do not change during the time evolution, and (1) becomes elongated in one direction while the other stay the same. This is the well known radial orbit instability. On the other hand figure 2 shows a more original property. Having no significant difference in this analysis, (2) and (3) have been merged in the same histogram. In these plots, we make the histogram of the random variable $[g, L^2]$ calculated for each star. The δt corresponding to the δf associated to the perturbation generator g, is arbitrarily choosen equal to $T_{dyn}/10$. The effect seen on this figure is clear, in the stable case $(2),(3)$, $[g, L^2]$ is symetric and highly peaked around the zero preserving value. On the other hand, in the unstable (1) case, we note a non-symetry of the histogram which is moreover less peaked than the previous. The effect, although not very inportant, exists according to analytical predictions.

4 Discussion

As we said in the introduction, the objective of this work is not to see the well known radial orbit instabillity. Our goal is more concentrated on the mechanism of this phenomenon. Basing ourself on recent analytical statements, we have tried to show that the stability susceptibility of a spherical collisionless stellar system is governed by his ability to receive preserving perturbations. However, there exists a criterion for this stability (see eq.(7)). But as all empiric results, it has some interpretation problems, and it is sometimes contested in its globality (Palmer et al,1991). Our study of $[g, L^2]$ histogram is more complicated than the previous analysis but it has a solid analytical result behind it, and we are able to undestand why the system is stable or not.

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